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The Finite Difference Methods for $\emptyset^4$–Nonlinear Klein Gordon Equation

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Abstract:- $\emptyset^4$ Klein Gordon equation has been solved numerically by using fully implicit finite difference method (FIFDM) and exponential finite difference method (ExpFDM) and we found that both methods can solve this kind of problems, example showed that fully implicit method is more a accurate than exponential finite difference method.

Keywords: - *Klein Gordon Equation, exponential finite difference method, fully implicit method.*

I. INTRODUCTION

Partial differential equations arise frequently in the formulation of fundamental laws of nature and in the mathematical analysis of a wide variety of problems in applied mathematics, mathematical physics, and engineering science. This subject plays a fundamental role in modern mathematical sciences, especially in physics, geometry, and analysis. Many problems of physical concern are described by partial differential equations with appropriate initial and/or boundary conditions [1]. The search for a new mathematical algorithm to discover the exact solutions or approximate solutions of nonlinear partial differential equations (PDEs) is an important and essential task in nonlinear science. One of the traditional techniques to find an approximate solution for the given problem is the finite difference method. The short history of the finite difference method starts with the 1930s. Even though some ideas may be traced back further, we begin the fundamental theoretical paper by Courant, Friedrichs and Lewy (1928) on the solutions of the problems of mathematical physics by finite differences. (Thomee 1999).[2]

II. INDENTATIONS AND EQUATIONS

II.1 Mathematical Model

The Klein-Gordon equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u - f(u) \qquad (1)$$

was named after the physicists Oskar Klein and Walter Gordon, who in 1926 proposed that it describes relativistic electrons. Some other authors make the similar claims in the same year. The equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves. The equation is found in his notebooks from late 1925, before he made the discovery of the equation that now bears his name. He rejected it because he couldn't make it fit data (the equation doesn't take in account the spin of the electron), the way he found his equation was by making simplifications in the Klein Gordon equation. In 1927, soon after the Schrödinger equation was introduced, Vladimir Fock wrote an article about its generalization for the case of magnetic fields, where forces were dependent on velocity, and independently derived this equation. Both Klein and Fock used Kaluza and Klein's method. [3]

Fiore et al. (2005) gave arguments for the existence of exact travelling wave solutions of a perturbed sine Gordon equation on the real line or on the circle and classified them [4].

When $f(u) = mu - \varepsilon u^3$ then equation (1) is called $\emptyset^4$–nonlinear klein Gordon equation ($\emptyset^4$ equation)[5]

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^3 \qquad (2)$$

or [11]

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - mu + \varepsilon u^3 \qquad (3)$$

With initial and boundary conditions [12]

$u(x,0) = f(x)$ and $\frac{\partial u(x,0)}{\partial t} = 0 \quad ,t \geq 0 \ \ , 0 < x < 2\pi$

$u(0,t) = u(2\pi,t) = 0$

Equation (3) arises in quantum field theory with m denoting mass and ε is coupling constant [6,7]. The φ^4 equation has become an important subject because of its numerous applications in condensed matter physics. It

describes, for example, structural phase transitions in ferroelectric and ferromagnetic materials, topological excitations in quasi one dimensional system like biological macromolecules and hydrogen chains, or polymers, etc. Its simplest localized solutions so-called "kinks" which are related to the motion of the aforementioned topological excitations, e.g., domain walls in second order phase transitions, or polymerization mismatches. A more realistic modeling of physical situation in condensed matter physics often requires the inclusion of perturbations of different types like thermal noise and time or spatial dependent potential fluctuations [8].
The φ^4 equation was first proposed by Aubry , Krumhansl and Schrieffer in 1975 and 1976, to describe displacive and order-disorder transitions in solids, mainly magnetic compounds [9]. Manna and Merle (1997) used multiple- sale perturbation theory. They showed that a nonlinear (quadratic) Klein – Gordon type equation substitutes in a short- wave analysis the ubiquitous Korteweg-de Vries equation of long-wave approach. Dmitriev et. al. (2006) discussed some discrete φ^4 equations free of the peierls-Nabarro barrier and identified for them the full space of available static solutions, including those derived recently in physics but not limited to them [1].

II.2 **Derivative of Fully Implicit Method for $\varnothing^4$ Equation**

In this method, we evaluate the unknown function $u_{p,q+1}$ at t_{q+1} from the known function $u_{p,q}$ at t_q, as shown in figure (1):

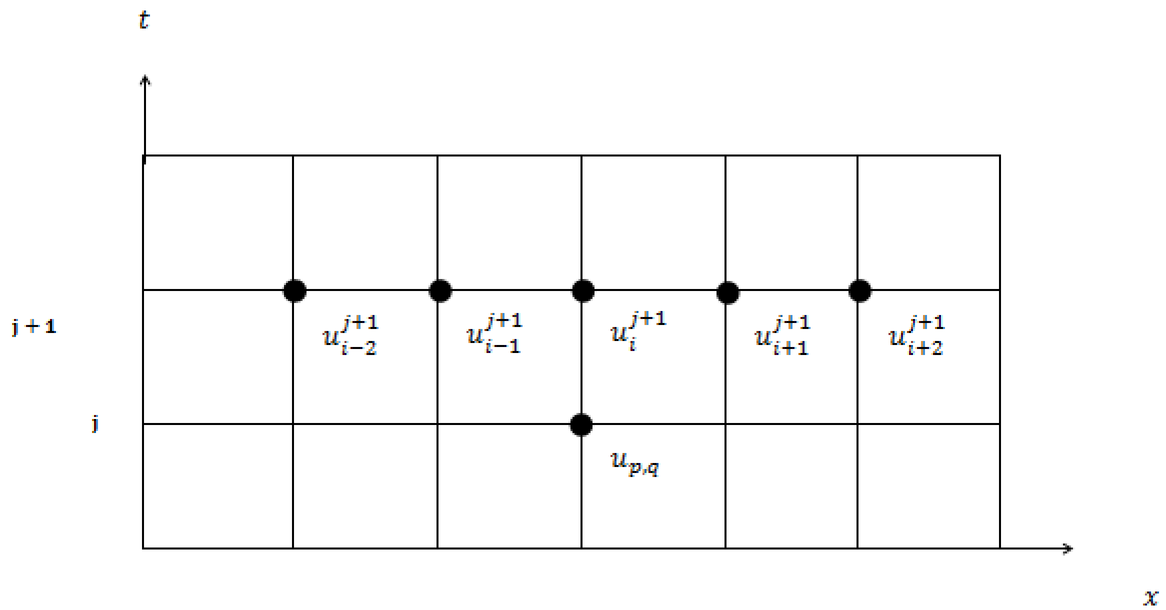

Figure (1): The mesh of the fully implicit method

Let the coordinate (x, t) of the grid points be $x = ph, t = qk$ Where p, q are integers. Denote the values of u at these mesh points by
$u(ph, qk) = u_{p,q}$

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{p,q} = \frac{u(p+1,q)-2u(p,q)+u(p-1,q)}{h^2} \qquad (4)$$

Similarly

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{p,q} = \frac{u(p,q+1)-2u(p,q)+u(p,q-1)}{k^2} \qquad (5)$$

Then, for derivation fully implicit method, we substitute (4) and (5) at (j+1), as follows:

$$\frac{u_{p,q+1} - 2u_{p,q} + u_{p,q-1}}{k^2} = \left(\frac{u_{p+1,q+1} - 2u_{p,q+1} + u_{p-1,q+1}}{h^2}\right) - mu_{p,q+1} + \xi u_{p,q+1}^3$$

$$u_{p,q+1} - 2u_{p,q} + u_{p,q-1} = ru_{p+1,q+1} - 2ru_{p,q+1} + ru_{p-1,q+1} - k^2 mu_{p,q+1} + k^2 \xi u_{p,q+1}^3$$

$$-ru_{p+1,q+1} + u_{p,q+1} + 2ru_{p,q+1} - ru_{p-1,q+1} + k^2 mu_{p,q+1} - k^2 \xi u_{p,q+1}^3 = 2u_{p,q} - u_{p,q-1}$$

$$-ru_{p-1,q+1} + (1 + 2r + k^2m - k^2 \varepsilon u_{p,q+1}^2)u_{p,q+1} - ru_{p+1,q+1} = 2u_{p,q} - u_{p,q-1} \qquad (6)$$

Recall that Taylor's formula of order two is:

$$u(x_p, k) = u(x_p, 0) + u_t(x_p, 0)k + \frac{u_{tt}(x_p, 0)k^2}{2} + O(k^3) \qquad (7)$$

Now, substitute (1) in (7), yields

$$u(x_p, k) = f_p + \frac{k^2}{2}(u_{xx} - mu + \varepsilon u^3) + O(k^3) \qquad (8)$$

Now, we use (8) to find the second row:

$$u_{p,2} - (\frac{k^2}{2})\left(\frac{u_{p+1,2} - 2u_{p,2} + u_{p-1,2}}{h^2}\right) + (\frac{k^2}{2})mu_{p,2} - (\frac{k^2}{2})\varepsilon u_{p,2}^3 = u_{p,1}$$

$$-\frac{r}{2}u_{p-1,2} + (1 + r + \frac{k^2}{2}m - \frac{k^2}{2}\varepsilon u_{p,2}^2)u_{p,2} - \frac{r}{2}u_{p+1,2} = u_{p,1} \qquad (9)$$

II.3 Derivative of Exponential Method for $\emptyset^4$ Equation

The exponential finite-difference method was originally developed by Bhattachary [1] and used to solve one dimensional heat conduction in a solid slab [11,12]. It is also used to solve the Korteweg-de Vries equation [13], we use this method to obtain the numerical approximation for Klein Gordon equation, which is given in(1).we start deriving the method by assuming that F (u) denote to any continuous differential function, and then by using chain rule, we have the following:

$$\frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial t^2} = \frac{\partial F}{\partial u}(u_{xx} - mu + \varepsilon u^3)$$

$$\frac{\partial}{\partial u}\left(\frac{\partial F \partial^2 u}{\partial u \partial t^2}\right) = \frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u}(u_{xx} - mu + \varepsilon u^3)\right)$$

$$\frac{\partial^2 F \partial^2 u}{\partial u^2 \partial t^2} = \left(\frac{\partial^2 F}{\partial u^2}(u_{xx} - mu + \varepsilon u^3)\right)$$

$$\frac{\partial^2 F}{\partial t^2} = F''(u)(u_{xx} - mu + \varepsilon u^3) \qquad (10)$$

Using the usual forward difference replacement to $\frac{\partial^2 F}{\partial t^2}$ we obtain the finite difference representation of Eq. (10) we get:

$$F(u_{i,j+1}) - 2F(u_{i,j}) + F(u_{i,j-1}) = k^2 F''(u)(u_{xx} - mu + \varepsilon u^3)$$

If we take $F(u) = \ln(u)$ and $F''(u) = \frac{-1}{u^2}$ then we obtained the exponential finite difference scheme as:

$$\ln(u_{i,j+1}) = 2\ln u_{(i,j)} - \ln u_{i,j-1} - \frac{k^2}{u_{(i,j)}{}^2}(u_{xx} - mu + \varepsilon u^3)$$

$$(u_{i,j+1}) = exp^{2\ln u_{(i,j)} - \ln u_{i,j-1} + (\frac{-k^2}{u_{(i,j)}{}^2}(u_{xx} - mu + \varepsilon u^3)}$$

let $r = \frac{k^2}{h^2}$,we get:

$$(u_{i,j+1}) = (u_{(i,j)}{}^2)(u_{i,j-1})^{-1} exp^{\frac{-r}{u_{(i,j)}{}^2}((u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{k^2 m}{u_{i,j}} - k^2 \varepsilon u_{i,j})}$$

$$u_{i,j+1} = \frac{u_{(i,j)}{}^2}{u_{i,j-1}} exp^{\frac{-r}{u_{(i,j)}{}^2}((u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{k^2 m}{u_{i,j}} - k^2 \varepsilon u_{i,j}}$$

$$u_{i,j+1} = \frac{u_{(i,j)}{}^2}{u_{i,j-1}} exp^{\frac{-r}{u_{(i,j)}{}^2}((u_{i+1,j} + u_{i-1,j}) + \frac{2r}{u_{i,j}} + \frac{k^2 m}{u_{i,j}} - k^2 \varepsilon u_{i,j}} \qquad (11)$$

Similar to the pervious methods, we use the Taylor series (7) to determine the second row and obtain:

$$u_{p,2} = u_{p,1} + \frac{k^2}{2}\left(\left(\frac{u_{p+1,1} - 2u_{p,1} + u_{p-1,1}}{h^2} - mu_{p,1} + \varepsilon u_{p,1}{}^3\right)\right.$$

$$u_{p,2} = u_{p,1} + \frac{r}{2}(u_{p+1,1} + u_{p-1,1}) - ru_{p,1} - \frac{k^2}{2}mu_{p,1} + \frac{k^2}{2}\varepsilon u_{p,1}{}^3$$

$$u_{p,2} = u_{p,1} + \frac{r}{2}\left((u_{p+1,1} + u_{p-1,1}) - ru_{p,1} - k^2 mu_{p,1}/2 + k^2 \varepsilon u_{p,1}{}^3/2\right)$$

$$u_{p,2} = (2 - 2r - k^2 m + k^2 \varepsilon u_{p,1}{}^2)u_{p,1} + r\left((u_{p+1,1} + u_{p-1,1})\right) \qquad (12)$$

III. FIGURES AND TABLES

III.1 Numerical Example

The following example solved numerically to illustrate efficiency of the presented methods.

Example *:* [14]

$$\frac{\partial^2 u}{\partial t^2} = -\alpha\frac{\partial^2 u}{\partial x^2} - \beta u - \gamma u^3 \qquad ,t > 0$$

With the initial conditions

$$u(x,0) = B\tan(Kx), u_t = BcK\,sec^2(Kx) \qquad , -1 \le x \le 1$$

We take $\alpha = -2.5,\ \beta = 1,\ \gamma = 1.5.$

Where $B = \sqrt{\frac{\beta}{\gamma}}$ and $K = \sqrt{\frac{-\beta}{2(\alpha+c^2)}}$

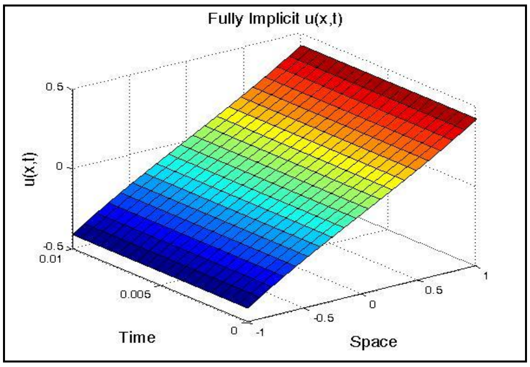

Fig. (2) Space-Time graph of fully implicit solution to 0<t<0.01 and **-**1<x<1 with h=0.105263157894737

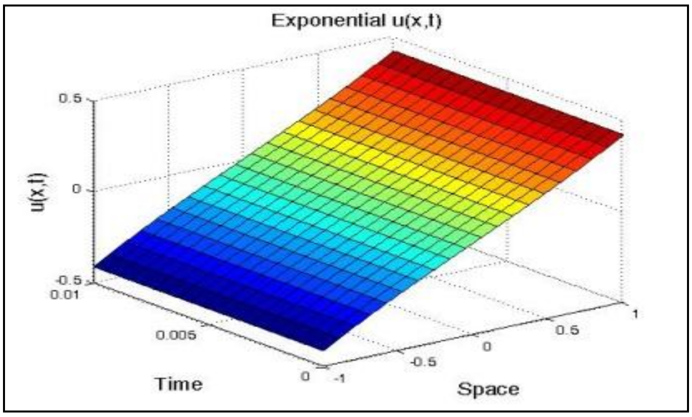

Fig. (3) Space-Time graph of exact solution to 0<t<0.01 and **-**1<x<1 with h=0.105263157894737

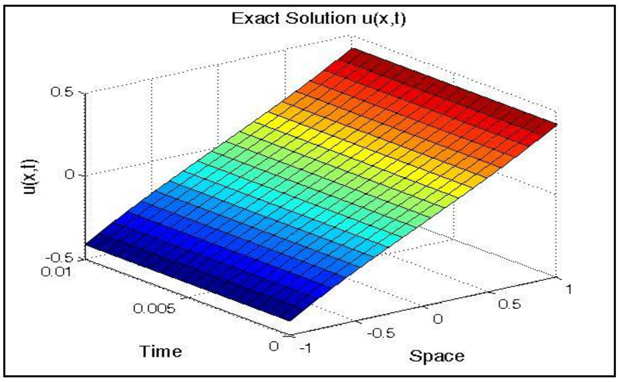

Fig. (4) Space-Time graph of Exponential solution to 0<t<0.01 and **-**1<x<1 with h=0.105263157894737

Table (1) comparison exact with fully implicit and exponential

Exact t=10	**Fully Implicit t=10**	**Exponential t=10**
-0.415048325818363	-0.415048325818363	-0.415048325818363
-0.365277881508473	-0.365160529736901	-0.365280203828274
-0.317670922704520	-0.317670559209555	-0.317673354402609
-0.271868804987755	-0.271868507872998	-0.271871393450315
-0.227556818695303	-0.227556568822620	-0.227559637169243
-0.184454716348066	-0.184454512977609	-0.184457883039865
-0.142309026690262	-0.142308869214878	-0.142312756563446
-0.100886681334808	-0.100886569280600	-0.100891442373613
-0.059969595059594	-0.059969528086989	-0.059976814870891
-0.019349919766315	-0.019349897670325	-0.019368155435091
0.021174254684441	0.021174231974307	0.021192412127243
0.061802945529073	0.061802877949001	0.061807831330644
0.102738241949429	0.102738129301538	0.102740721843501
0.319771946441709	0.319771582558915	0.319771774387514
0.417345261791338	0.417345261791338	0.417345261791338

IV. CONCLUSION

We saw that Fully implicit method is much more accurate than Exponential finite difference method for solving ϕ^4 Klein Gordon Equation and this kind of models as shown in figures (2-4) and table (1).

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